Basic Math Formulas, Conversion Tables, & Other Study Aids

For High School And College Students

Compilation by Wizard John
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O Icarus...

I ride high...
With a whoosh to my back
And no wind to my face,
Folded hands
In quiet rest—
Watching...O Icarus...
The clouds glide by,
Their fields far below
Of gold-illumed snow,
Pale yellow, floating moon
To my right—evening sky.

And Wright...O Icarus...
Made it so—
Silvered chariot streaking
On tongues of fire leaping—
And I will soon be sleeping
Above your dreams...J. Sparks 2001

100th Anniversary of
Powered Flight:
1903—2003

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1) Allied Health Conversion Factors

Apothecary, Household and Metric Systems

- All three systems have rough volume equivalents.
- Since the household system is a volume-only system, the Weight Exchange Table does not include household equivalents.
- Common discrepancies that are still considered correct are shown in *italics*.

1.1 Volume Conversion Table				
Apothecary	**Apothecary**	**Household**	**Household**	**Metric**
1minim		1drop	1gtt	
16minims				1mL (cc)
60minims	1fluidram	60gtts	1tsp	5mL (cc) *or 4mL*
4fluidrams	0.5fluidounce	3tsp	1tbsp	15mL (cc)
8fluidrams	1fluidounce	2tbsp		30mL (cc)
	8fluidounces	1cup		240mL (cc)
	16fluidounces	2cups	1pint	500mL (cc) *or 480mL*
	32fluidounces	2pints	1quart	1000mL (cc) *or 960mL*

1.2 Weight Conversion Table		
Apothecary	**Apothecary**	**Metric**
1grain		60mg *or 64mg*
15grains		1g
60grains	1dram	4g
8drams	1ounce	32g
12ounces	1pound	384g

1.3 Temperature Conversion Formulas

A) Fahrenheit to Celsius: $C = \dfrac{F - 32}{1.8}$

B) Celsius to Fahrenheit: $F = 1.8C + 32$

2) Algebra

2.1 Field Axioms

The field axioms *decree* the fundamental operating properties of the real number system and provide the basis for all advanced operating properties in mathematics.

Let $a, b \,\&\, c$ be any three real numbers		
Properties	**Addition**	**Multiplication**
Closure	$a+b$ is a unique real number	$a \cdot b$ is a unique real number
Commutative	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
Identity	$0 \Rightarrow a+0=a$	$1 \Rightarrow a \cdot 1 = a$
Inverse	$a \Rightarrow a+(-a)=0$ $\Rightarrow (-a)+a=0$	$a \neq 0 \Rightarrow a \cdot \frac{1}{a} = 1$ $\Rightarrow \frac{1}{a} \cdot a = 1$
Distributive *or Linking Property*	$a \cdot (b+c) = a \cdot b + a \cdot c$	
Note: $ab = a(b) = (a)b$ *are alternate representations of* $a \cdot b$		

2.2 Subtraction and Division

1. Definitions

   Subtraction: $a - b \equiv a + (-b)$

   Division: $a \div b \equiv a \cdot \dfrac{1}{b}$

2. Alternate representation of $a \div b$: $a \div b = \dfrac{a}{b}$

3. Division Properties of Zero

Zero in numerator: $a \neq 0 \Rightarrow \frac{0}{a} = 0$

Zero in denominator: $\frac{a}{0}$ is undefined!

Zero in both: $\frac{0}{0}$ is undefined!

2.3 Rules for Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions with $b \neq 0$ and $d \neq 0$.

1. Equality: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$
2. Equivalency: $c \neq 0 \Rightarrow \frac{a}{b} = \frac{ac}{bc} = \frac{ca}{cb} = \frac{ac}{cb} = \frac{ca}{bc}$
3. Addition (like denominators): $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$
4. Addition (unlike denominators): $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$
5. Subtraction (like denominators): $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$
6. Subtraction (unlike denominators): $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{cb}{bd} = \frac{ad-cb}{bd}$
7. Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
8. Division: $c \neq 0 \Rightarrow \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$
9. Reduction of Complex Fraction: $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}$
10. Placement of Sign: $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

2.4 Rules for Exponents

1. Addition: $a^n a^m = a^{n+m}$
2. Subtraction: $\frac{a^n}{a^m} = a^{n-m}$

3. Multiplication: $(a^n)^m = a^{nm}$

4. Distributed over a Simple Product: $(ab)^n = a^n b^n$

5. Distributed over a Complex Product: $(a^m b^p)^n = a^{mn} b^{pn}$

6. Distributed over a Simple Quotient: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

7. Distributed over a Complex Quotient: $\left(\dfrac{a^m}{b^p}\right)^n = \dfrac{a^{mn}}{b^{pn}}$

8. Definition of Negative Exponent: $\dfrac{1}{a^n} \equiv a^{-n}$

9. Definition of Radical Expression: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$

10. When No Exponent is Present: $a = a^1$

11. Zero Exponent: $a^0 = 1$

2.5 Factor Formulas

1. Simple Common Factor: $ab + ac = a(b+c) = (b+c)a$
2. Grouped Common Factor:

$$ab + ac + db + dc = (b+c)a + d(b+c) =$$
$$(b+c)a + (b+c)d = (b+c)(a+d)$$

3. Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
4. Sum of Squares: $a^2 + b^2$ *is not factorable!*
5. Perfect Square: $a^2 \pm 2ab + b^2 = (a \pm b)^2$
6. General Trinomial: $x^2 + (a+b)x + ab = (x+a)(x+b)$
7. Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
8. Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
9. Power Reduction to an Integer: $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$
10. Power Reduction to a Radical: $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$
11. Power Reduction to an Integer plus a Radical:

$$a^2 + ab + b^2 = (a + \sqrt{ab} + b)(a - \sqrt{ab} + b)$$

2.6 Laws of Equality

Let $A = B$ be an algebraic equality and C be any quantity.

1. Addition: $A + C = B + C$
2. Subtraction: $A - C = B - C$
3. Multiplication: $A \cdot C = B \cdot C$

4. Division: $\frac{A}{C} = \frac{B}{C}$ provided $C \neq 0$

5. Exponent:: $A^n = B^n$ provided n is an integer

6. Reciprocal: $\frac{1}{A} = \frac{1}{B}$ provided $A \neq 0, B \neq 0$

7. Zero Product Property: $A \cdot B = 0$ if and only if $A = 0$ or $B = 0$

2.7 Rules for Radicals

1. Basic Definitions: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$ and $\sqrt[2]{a} \equiv \sqrt{a} \equiv a^{\frac{1}{2}}$
2. Complex Radical: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
3. Associative: $(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$
4. Simple Product: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
5. Simple Quotient: $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
6. Complex Product: $\sqrt[n]{a}\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$
7. Complex Quotient: $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \sqrt[nm]{\frac{a^m}{b^n}}$
8. Nesting: $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$
9. Rationalization Rules for $n > m$

   Numerator: $\frac{\sqrt[n]{a^m}}{b} = \frac{a}{b\sqrt[n]{a^{n-m}}}$

   Denominator: $\frac{b}{\sqrt[n]{a^m}} = \frac{b\sqrt[n]{a^{n-m}}}{a}$

2.8 Rules for Logarithms

1. Definition of Logarithm to Base $b > 0$: $y = \log_b x$ if and only if $b^y = x$
2. Logarithm of the Same Base: $\log_b b = 1$
3. Logarithm of One: $\log_b 1 = 0$
4. Logarithm of the Base to a Power: $\log_b b^p = p$
5. Base to the Logarithm: $b^{\log_b p} = p$
6. Notation for Logarithm Base 10: $Logx = \log_{10} x$
7. Notation for Logarithm Base e: $\ln x = \log_e x$

8. Product: $\log_b(MN) = \log_b N + \log_b M$

9. Quotient: $\log_b\left(\dfrac{M}{N}\right) = \log_b N - \log_b M$

10. Power: $\log_b N^p = p\log_b N$

11: Change of Base Formula: $\log_b N = \dfrac{\log_a N}{\log_a b}$

2.9 Complex Numbers

1. Properties of the imaginary unit i: $i^2 = -1 \Rightarrow i = \sqrt{-1}$
2. Definition of complex number: Numbers of the form $a + bi$ where a, b are real numbers
4. Definition of Complex Conjugate: $\overline{a + bi} = a - bi$
5. Definition of Modulus: $|a + bi| = \sqrt{a^2 + b^2}$
6. Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
7. Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
8. Multiplication: $(a + bi)(c + di) = ac + (ad + bc)i + bdi^2 = ac - bd + (ad + bc)i$
9: Division:

$$\frac{a+bi}{c+di} = \frac{(a+bi)\overline{(c+di)}}{(c+di)\overline{(c+di)}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$\frac{(ac+bd)+(bc-ad)i}{c^2-d^2} = \frac{ac+bd}{c^2-d^2} + \left(\frac{bc-ad}{c^2-d^2}\right)i$$

2.10 Quadratic Equations and Functions

Let $ax^2 + bx + c = 0, a \neq 0$ be a quadratic equation

1. Quadratic Formula for Solutions x: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. Solution Discriminator: $b^2 - 4ac$

   Two real solutions: $b^2 - 4ac > 0$
   One real solution: $b^2 - 4ac = 0$
   Two complex solutions: $b^2 - 4ac < 0$
3. Solution when $a = 0 \,\&\, b \neq 0$: $bx + c = 0 \Rightarrow x = \dfrac{-c}{b}$
4. Definition of Quadratic-in-Form Equation:
   $aw^2 + bw + c = 0$ where w is a algebraic expression
5. Definition of Quadratic Function: $f(x) = ax^2 + bx + c$

6. Axis of Symmetry for Quadratic Function: $x = \frac{-b}{2a}$

7. Vertex for Quadratic Function: $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$

2.11) Theory of Polynomial Equations

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ be a polynomial written in standard form.

Eight Basic Theorems

1. Fundamental Theorem of Algebra: Every polynomial $P(x)$ of degree $N \geq 1$ has at least one solution x_0 for which $P(x_0) = 0$. This solution may be real or complex (i.e. has the form $a + bi$).

2. Numbers Theorem for Roots and Turning Points: If $P(x)$ is a polynomial of degree N, then the equation $P(x) = 0$ has up to N real solutions or *roots*. The equation $P(x) = 0$ has exactly N roots if one counts complex solutions of the form $a + bi$. Lastly, the graph of $P(x)$ will have up to $N - 1$ turning points (which includes both relative maxima and minima).

3. Real Root Theorem: If $P(x)$ is of odd degree having all real coefficients, then $P(x)$ has at least one real root.

4. Rational Root Theorem: If $P(x)$ has all integer coefficients, then any rational roots for the equation $P(x) = 0$ must have the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0 and q is a factor of the lead coefficient a_n. *Note: This result is used to form a rational-root possibility list.*

5. Complex Conjugate Pair Root Theorem: Suppose $P(x)$ has all real coefficients. If $a + bi$ is a root for $P(x)$ with $P(a + bi) = 0$, then $P(a - bi) = 0$.

6. Irrational Surd Pair Root Theorem: Suppose $P(x)$ has all rational coefficients. If $a + \sqrt{b}$ is a root for $P(x)$ with $P(a + \sqrt{b}) = 0$, then $P(a - \sqrt{b}) = 0$.

7. Remainder Theorem: If $P(x)$ is divided by $(x - c)$, then the remainder R is equal to $P(c)$. *Note: this result is extensively used to evaluate a given polynomial $P(x)$ at various values of x.*

8. Factor Theorem: If c is any number with $P(c) = 0$, then $(x - c)$ is a factor of $P(x)$. This means $P(x) = (x - c) \cdot Q(x)$ where $Q(x)$ is a new, reduced polynomial having degree one less than $P(x)$. The converse is also true $P(x) = (x - c) \cdot Q(x) \Rightarrow P(c) = 0$.

Three Advanced Theorems

9. Root Location Theorem: Let (a,b) be an interval on the x axis with $P(a) \cdot P(b) < 0$. Then there is a value $x_0 \in (a,b)$ such that $P(x_0) = 0$.

10. Root Bounding Theorem: Divide $P(x)$ by $(x-d)$ to obtain $P(x)=(x-d)\cdot Q(x)+R$. Case $d>0$: If both R and all the coefficients of $Q(x)$ are positive, then $P(x)$ has no root $x_0>d$. Case $d<0$: If the roots of $Q(x)$ alternate in sign—with the remainder R "in sync" at the end—then $P(x)$ has no root $x_0<d$. *Note: Coefficients of zero can be counted either as positive or negative—which ever way helps in the subsequent determination.*

11. Descartes' Rule of Signs: Arrange $P(x)$ in standard order as shown in the title bar. The number of positive real solutions equals the number of coefficient sign variations or that number decreased by an even number. Likewise, the number of negative real solutions equals the number of coefficient sign variations in $P(-x)$ or that number decreased by an even number.

2.12 Determinants and Cramer's Rule

1. Determinant Expansions

Two by Two: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Three by Three: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

2. Cramer's Rule for a Two by Two Linear System

Given $\begin{matrix} ax+by=e \\ cx+dy=f \end{matrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Then $x=\dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and $y=\dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

4. Cramer's Rule for a Three by Three Linear System

Given $\begin{matrix} ax+by+cz=j \\ dx+ey+fz=k \\ gx+hy+iz=l \end{matrix}$ with $D=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$

Then $x=\dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{D}, y=\dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{D}, z=\dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{D}$

2.13 Binomial Theorem

1. Definition of $n!$ where n is a positive integer:
   $n! = n(n-1)(n-2)...1$
2. Special Factorials: $0! = 1$ and $1! = 1$
3. Combinatorial Symbol: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$
4. Summation Symbols:

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + a_3 + a_4 + ... + a_n$$

$$\sum_{i=k}^{n} a_i = a_k + a_{k+11} + a_{k+2} + a_{k+3} ... + a_n$$

5. Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$
6. Sum of Binomial Coefficients when $a = b = 1: \sum_{i=0}^{n} \binom{n}{i} = 2^n$

2.14 Geometric Series

1. Definition: $\sum_{i=0}^{n} ar^i$ where r is the common ratio
2. Summation Formula for $\sum_{i=0}^{n} ar^i$: $\sum_{i=0}^{n} ar^i = \dfrac{a(1-r^{n+1})}{1-r}$
3. Summation for Infinite Number of Terms Provided $0 < r < 1$

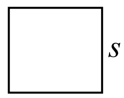
$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

3) Geometry

3.1 Planar Areas and Perimeters

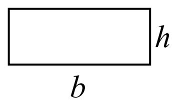
A is the planar area, P is the perimeter

1. Square:



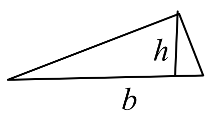
$A = s^2$ and $P = 4s$; s is the length of a side.

2. Rectangle:



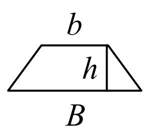
$A = bh$ and $P = 2b + 2h$; b & h are the base and height.

3. Triangle:



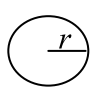
$A = \frac{1}{2}bh$; b & h are the base and altitude.

4. Trapezoid:



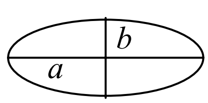
$A = \frac{1}{2}(B + b)h$; B & b are the two parallel bases,
and h is the altitude.

5. Circle:



$A = \pi r^2$ and $P = 2\pi r$; r is the radius.

6. Ellipse:

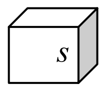


$A = \pi ab$; a & b are the half lengths of the
major & minor axes.

3.2 Solid Volumes and Surface Areas

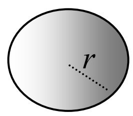
A is total surface area, V is the volume

1. Cube:



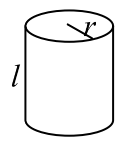
$A = 6s^2$ and $V = s^3$; s is the length of a side.

2. Sphere:



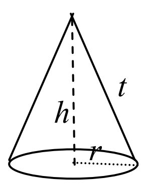
$A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$; r is the radius.

3. Cylinder:



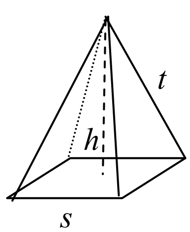
$A = 2\pi r^2 + 2\pi rl$ and $V = \pi r^2 l$;
r & l are the radius and length.

4. Cone:



$A = \pi r^2 + 2\pi rt$ and $V = \frac{1}{3}\pi r^2 h$;
r & t & h are radius, slant height, and altitude.

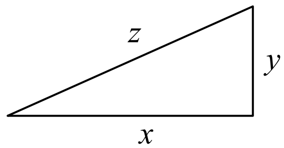
5. Pyramid (square base):



$A = s^2 + 2st$ and $V = \frac{1}{3}s^2 h$;
s & t & h are side, slant height, and altitude.

3.3 Pythagorean Theorem

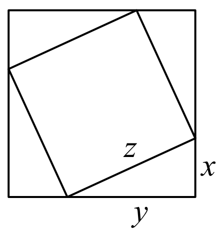
Theorem Statement: Given a right triangle with one side of length x, a second side of length y, and hypotenuse of length z.



Then: $z^2 = x^2 + y^2$

1. **Algebraic Proof**: Construct a big square by bringing together four congruent right triangles where each is a replicate of the triangle shown above.

$A = (x+y)^2$



$A = z^2 + 4\left(\dfrac{xy}{2}\right)$

The area of the big square is given by $A = (x+y)^2$, or equivalently by $A = z^2 + 4\left(\dfrac{xy}{2}\right)$.

Equating:

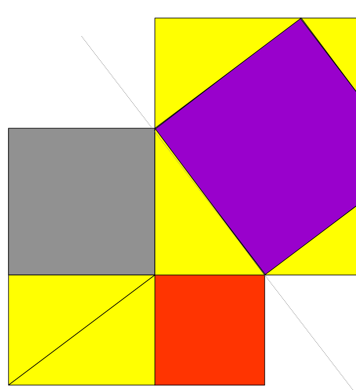
$$(x+y)^2 = z^2 + 4\left(\frac{xy}{2}\right) \Rightarrow$$

$$x^2 + 2xy + y^2 = z^2 + 2xy \Rightarrow.$$

$$x^2 + y^2 = z^2 \Rightarrow$$

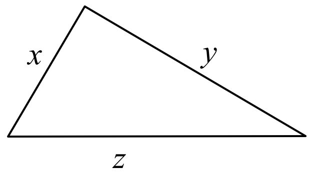
$$z^2 = x^2 + y^2 \therefore$$

2. **Pre-Algebraic (Totally Visual) Proof**:



3.4 Heron's Formula for Triangular Area

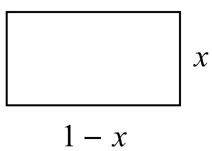
Let $p = \frac{1}{2}(x + y + z)$ be the semi-perimeter of a general triangle



Then: $A = \sqrt{p(p-x)(p-y)(p-z)}$

3.5 Golden Ratio

Definition: Let $p = 1$ be the semi-perimeter of a rectangle whose base and height are in the proportion shown. This proportion defines the Golden Ratio.



$$\frac{x}{1-x} = \frac{1-x}{1}$$

Golden Ratio

Solving: $x = 0.3819$ and $1 - x = 0.6181$

3.6 Distance and Line Formulas

Let (x_1, y_1) and (x_2, y_2) be two points where $x_2 > x_1$.

1. Distance Formula: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Midpoint Formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
3. Slope of Line: $m = \frac{y_2 - y_1}{x_2 - x_1}$
4. Point/Slope Equation of Line: $y - y_1 = m(x - x_1)$
5. General Form Of Equation of Line: $Ax + By + C = 0$
6. Slope/Intercept Equation of Line: $y = mx + b$

   x and y Intercepts: $\frac{-b}{m}$ and b
7. Slope of Parallel Line: m
8. Slope of Line Perpendicular to a Given Line of Slope m : $\frac{-1}{m}$

3.7 Conic Section Formulas

1. Circle of Radius r Centered at (h,k):

$$(x-h)^2 + (y-k)^2 = r^2$$

2. Ellipse Centered at (h,k):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

If $a > b$, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 - b^2$.

If $b > a$, the two foci are on the line $x = h$ and are given by $(h,k-c)$ & $(h,k+c)$ where $c^2 = b^2 - a^2$.

3. Hyperbola Centered at (h,k):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

When $\frac{(x-h)^2}{a^2}$ is to the left of the minus sign, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 + b^2$.

When $\frac{(y-k)^2}{b^2}$ is to the left of the minus sign, the two foci are on the line $x = h$ and are given by $(h,k-c)$ & $(h,k+c)$ where $c^2 = b^2 + a^2$.

4. Parabola with Vertex at (h,k) and Focal Length p:

$$(y-k)^2 = 4p(x-h) \text{ or } (x-h)^2 = 4p(y-k)$$

$(y-k)^2$: the focus is $(h+p,k)$ and the directrix is given by the line $x = h - p$.

$(x-h)^2$: the focus is $(h,k+p)$ and the directrix is given by the line $y = k - p$.

4) Money and Finance

P is the amount initially borrowed or deposited.
A is the total amount gained or owed.
r is the annual interest rate.
i is the annual inflation rate.
α is an annual growth rate as in the growth rate of voluntary contributions to a fund.
r_{eff} is the effective annual interest rate.
t is the time period in years for an investment.
T is the time period in years for a loan.
N is the number of compounding periods per year.
M is the monthly payment.

4.1 Simple Interest

1. Interest alone: $I = \Pr T$
2. Total repayment over T: $R = P + \Pr T = P(1+rT)$
3. Monthly payment over $T : M = \dfrac{P(1+rT)}{12T}$

4.2 Simple Principle Growth and Decline

1. Compounded Growth: $A = P(1+\frac{r}{N})^{Nt}$
2. Continuous Growth: $A = Pe^{rt}$
3. Continuous Annual Inflation rate i: $A = Pe^{-it}$

4.3 Effective Interest Rates

1. N compounding periods per year: $r_{eff} = (1+\frac{r}{N})^{N} - 1$
2. Continuous interest: $r_{eff} = e^{r} - 1$
3. Known P, A, T: $r_{eff} = \sqrt[T]{\dfrac{A}{P}} - 1$

4.4 Continuous Interest IRA Growth Formulas

1. Annual deposit D: $A = \dfrac{D}{r}(e^{rt} - 1)$
2. Annual deposit D plus initial deposit $P : A = Pe^{rt} + \dfrac{D}{r}(e^{rt} - 1)$

3. Annual deposit D continuously growing according to $De^{\alpha t}$ plus initial deposit P:

$$A = Pe^{rt} + \frac{D}{r-\alpha}(e^{rt} - e^{\alpha t})$$

3. Replacement Formula: Continuous Interest to Compounded Interest

Replace e^{rt} with $(1+\frac{r}{N})^{Nt}$

4.5 Continuous Interest Mortgage Formulas

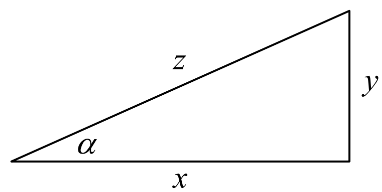
1. First month's Interest: $I_{1st} = \dfrac{rP}{12}$
2. Monthly payment: $M = \dfrac{\Pr e^{rT}}{12(e^{rT}-1)}$
3. Total repayment ($P+I$): $A = \dfrac{\Pr Te^{rT}}{e^{rT}-1}$
4. Total interest repayment: $I = P\left[\dfrac{rTe^{rT}}{e^{rT}-1} - 1\right]$

5. Replacement Formula: Continuous Principle Reduction to Monthly Principle Reduction

Replace e^{rT} with $(1+\frac{r}{12})^{12T}$

5) Trigonometry

5.1 Basic Definitions



Let the figure above be a right triangle with one side of length x, a second side of length y, and a hypotenuse of length z. The angle α is opposite the side of length. The six trigonometric functions—where each is a function of α —are defined as follows:

Arbitrary z	For $z = 1$
1. $\sin(\alpha) = \dfrac{y}{z}$	$\sin(\alpha) = y$
2. $\cos(\alpha) = \dfrac{x}{z}$	$\cos(\alpha) = x$
3. $\tan(\alpha) = \dfrac{y}{x}$	$\tan(\alpha) = \dfrac{y}{x}$
4. $\cot(\alpha) = \dfrac{x}{y}$	$\cot(\alpha) = \dfrac{x}{y}$
5. $\sec(\alpha) = \dfrac{z}{x}$	$\sec(\alpha) = \dfrac{1}{x}$
6. $\csc(\alpha) = \dfrac{z}{y}$	$\csc(\alpha) = \dfrac{1}{y}$

5.2 Definition-Based Identities

1. $\csc(\alpha) = \dfrac{1}{\sin(\alpha)}$
2. $\sec(\alpha) = \dfrac{1}{\cos(\alpha)}$
3. $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$
4. $\cot(\alpha) = \dfrac{\cos(\alpha)}{\sin(\alpha)}$
5. $\tan(\alpha) = \dfrac{1}{\cot(\alpha)}$

5.3 Pythagorean Identities

1. $\sin^2(\alpha) + \cos^2(\alpha) = 1$
2. $1 + \tan^2(\alpha) = \sec^2(\alpha)$
3. $1 + \cot^2(\alpha) = \csc^2(\alpha)$

5.4 Negative Angle Identities

1. $\sin(-\alpha) = -\sin(\alpha)$
2. $\cos(-\alpha) = \cos(\alpha)$
3. $\tan(-\alpha) = -\tan(\alpha)$
4. $\cot(-\alpha) = -\cot(\alpha)$

5.5 Sum and Difference Identities

1. $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
2. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
3. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
4. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
5. $\tan(\alpha + \beta) = \dfrac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$
6. $\tan(\alpha - \beta) = \dfrac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

5.6 Double Angle Identities

1. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
2. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$
   $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
3. $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$

5.7 Half Angle Identities

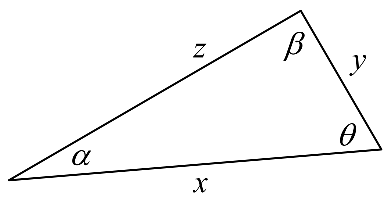
1. $\sin(\frac{\alpha}{2}) = \pm\sqrt{\dfrac{1 - \cos(\alpha)}{2}}$

2. $\cos(\frac{\alpha}{2}) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$

3. $\tan(\frac{\alpha}{2}) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}} = \frac{\sin(\alpha)}{1+\cos(\alpha)} = \frac{1-\cos(\alpha)}{\sin(\alpha)}$

5.8 General Triangle Formulas

Applicable to non-right triangles



1. Law of Sines: $\frac{y}{\sin(\alpha)} = \frac{x}{\sin(\beta)} = \frac{z}{\sin(\theta)}$

2. Law of Cosines:

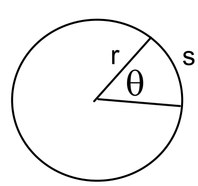
   a) $y^2 = x^2 + z^2 - 2xz\cos(\alpha)$
   b) $x^2 = y^2 + z^2 - 2yz\cos(\beta)$
   c) $z^2 = x^2 + y^2 - 2xy\cos(\theta)$

3. Area Formulas for a General Triangle:

   a) $A = \frac{1}{2}xz\sin(\alpha)$
   b) $A = \frac{1}{2}yz\sin(\beta)$
   c) $A = \frac{1}{2}xy\sin(\theta)$

5.9 Arc and Sector Formulas

1. Conversion: $180^0 = \pi$ radians
2. Arc length s : $s = r\theta$
3. Area of a Sector: $A = \frac{1}{2}r^2\theta$



5.10: Polar Form of Complex Numbers

1. $a+bi=r(\cos\theta+i\sin\theta)$ where $r=\sqrt{a^2+b^2}$, $\theta=Tan^{-1}\left[\dfrac{b}{a}\right]$
2. Statement of de Moivre's Theorem: $\left[r(\cos\theta+i\sin\theta)\right]^n=r^n(\cos[n\theta]+i\sin[n\theta])$
3. Definition of $re^{i\theta}$: $re^{i\theta}=r(\cos\theta+i\sin\theta)$
4. Euler's Famous Equality: $e^{i\pi}=-1$

6) Elementary Calculus

6.1 Basic Differentiation Rules

1. Limit Definition of: $f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$

2. Constant: $[k]' = 0$

3. Power: $[x^n]' = nx^{n-1}$, n can be <u>any exponent</u>

4. Coefficient: $[af(x)]' = af'(x)$

5. Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

6. Product: $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$

7. Quotient: $\left[\frac{f(x)}{g(x)} \right]' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

8. Chain: $[f(g(x))]' = f'(g(x))g'(x)$.

9. Inverse: $[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$

10. Generalized Power: $[\{f(x)\}^n]' = n\{f(x)\}^{n-1} f'(x)$;

    Again, n can be <u>any exponent</u>

6.2 Basic Antidifferentiation Rules

1. Constant: $\int k dx = kx + C$
2. Coefficient: $\int af(x)dx = a\int f(x)dx$
3. Power:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C, n = -1$$

4.Sum/Difference: $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

5. Parts: $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

6. Chain: $\int f'(g(x))g'(x)dx = f(g(x)) + C$

7. Generalized Power:

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$$

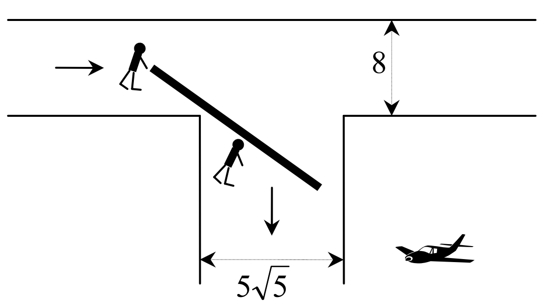
$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C, n = -1$$

6.3 The Fundamental Theorem of Calculus

Consider the definite integral $\int_a^b f(x)dx$, which can be thought of as a continuous addition process on the interval $[a,b]$, a process that sums millions upon millions of tiny quantities having the general form $f(x)dx$ from $x = a$ to $x = b$. Now, let $F(x)$ be any antiderivative for $f(x)$ where, by definition, we have that $F'(x) = f(x)$. Then, the summation process $\int_a^b f(x)dx$ can be evaluated by the alternative process $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.

6.4 Famous Girder Problem

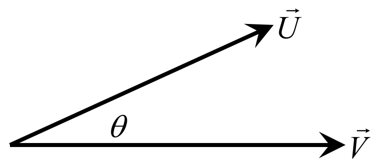
Two people at a construction site are rolling steel girders down a corridor 8 feet wide into a second corridor $5\sqrt{5}$ feet wide and perpendicular to the first corridor. What is the length of the longest girder that can be rolled from the first corridor into the second corridor and continued on its journey in the construction site? Assume the girder is of negligible thickness. *The above problem started to appear in calculus texts circa 1900 and is famous because of the way it thoroughly integrates the principles of plane geometry, algebra, and differential calculus. My experience as a teacher has been that "many try, but few succeed." Can you?*



7) Elementary Vector Algebra

7.1 Basic Definitions and Properties

Let $\vec{V} = < v_1, v_2, v_3 >$, $\vec{U} = < u_1, u_2, u_3 >$



1. $\vec{U} \pm \vec{V} = < u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 >$
2. $(\alpha)\vec{U} = < \alpha u_1, \alpha u_2, \alpha u_3 >$
3. $-\vec{U} = (-1)\vec{U}$
4. $[\vec{U}] = \sqrt{u_1^2 + u_2^2 + u_3^2}$
5. $\vec{0} = (0,0,0)$
6. Unit Vector Parallel to $\vec{V}$: $\dfrac{1}{[\vec{V}]}\vec{V}$
7. Two Parallel Vectors: $\vec{V} \,||\, \vec{U} \Rightarrow \exists c, st \vec{V} = (c)\vec{U}$

7.2 Dot Products

1. Definition of Dot Product: $\vec{U} \bullet \vec{V} = u_1 v_1 + u_2 v_2 + u_3 v_3$
2. Angle $\theta : \cos\theta = \dfrac{\vec{U} \bullet \vec{V}}{[\vec{U}][\vec{V}]}$
3. Orthogonal Vectors: $\vec{U} \bullet \vec{V} = 0$
4. Projection of $\vec{U}$ onto $\vec{V}$: $proj_{\vec{V}}(\vec{U}) = \left[\dfrac{\vec{U} \bullet \vec{V}}{[\vec{V}]^2}\right]\vec{V} = \left[\dfrac{\vec{U} \bullet \vec{V}}{[\vec{V}]}\right]\dfrac{\vec{V}}{[\vec{V}]} = \left[[\vec{U}]\cos\theta\right]\dfrac{\vec{V}}{[\vec{V}]}$

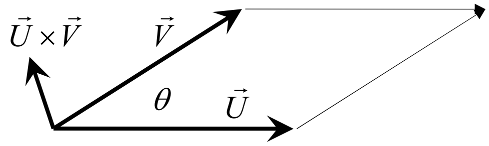
7.3 Cross Products

1. Definition of Cross Product: $\vec{U} \times \vec{V} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

2. Orientation of $\vec{U} \times \vec{V}$: Orthogonal to Both $\vec{U}$ and $\vec{V}$ i.e. $\vec{U} \bullet (\vec{U} \times \vec{V}) = \vec{V} \bullet (\vec{U} \times \vec{V}) = 0$

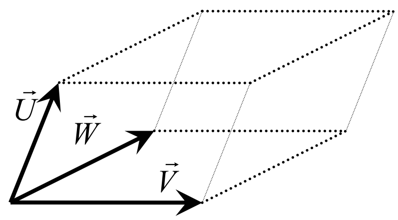
3. Area of parallelogram

$$[\vec{U} \times \vec{V}] = [\vec{U}][\vec{V}]\sin\theta$$



4. Interpretation of the Triple Scalar Product: $\vec{U} \bullet (\vec{V} \times \vec{W}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Volume of the Parallelepiped Below



7.4 Line and Plane Equations

1. Line parallel to $< a, b, c >$ and passing through (x_1, y_1, z_1).
   If (x, y, z) is a point on the line, then:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

2. Equation of Plane Normal to $< a, b, c >$ and Passing Through (x_1, y_1, z_1).
   If (x, y, z) is a point on the plane, then:

$$< a, b, c > \bullet < x - x_1, y - y_1, z - z_1 >= 0.$$

3. Distance D between a point & plane:
   If a point is given by (x_0, y_0, z_0) and $ax + by + cz + d = 0$ is a plane, then:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

8) Statistics

8.1 Basic Definitions and Concepts

1. A **set** is an aggregate of individual items—animate or inanimate.
2. An **element** of the set is a particular item in the set.
3. An **observation** associated with the element is any attribute of interest.
4. A **statistic** associated with the element is any measurement of interest.
   Any statistic is an observation, but not all observations are statistics.
5. **Statistics**: the science of drawing conclusions from the totality of observations.
6. A **population** is the totality of elements that one wishes to study by making observations.
7. A **sample** is that population subset that one has the resources to study.
8. **Random sample**: where all population elements have equal probability of access.

8.2 Basic Descriptors

Let a set consist of N elements where there has been observed one statistic of a similar nature for each element. The data set of all observed statistics is denoted by $\{x_1, x_2, x_3, ..., x_N\}$. Data sets can come from either populations or a samples. More than likely, the data set will be considered a sample and will be utilized to make predictions about a corresponding and much larger population.

Measures of data "centering" or central tendency

1. Sample Mean or Average $\bar{x}$: $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$
2. Population Mean or Average μ : $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$
2. Rank-Ordered Data Set: a re-listing of the statistics $\{x_1, x_2, x_3, ..., x_N\}$ in numerical order from smallest to largest
3. Median $\tilde{x}$: the statistic the middle of a rank-ordered data set
   $\tilde{x}$ is the actual middle statistic if an odd number of data points
   $\tilde{x}$ is the average of the two middle statistics if an even number of data points
4. Mode M : the data value or statistic that occurs most often.
5. Multi-modal Data Set: a data set with two or more modes

Measures of data "spread" or dispersion

1. Range $R : R = x_L - x_S$ where x_L is the largest data value in the data set
   and x_S is the smallest data value
2. Sample Standard Deviation $s : s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$. The Sample Variance is s^2
3. Population Standard Deviation $\sigma : \sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$. The Population Variance is σ^2.
4. Coefficient of Variation C_V : Sample $C_V = \dfrac{s}{\bar{x}}$, Population $C_V = \dfrac{\sigma}{\mu}$

8.3 Exploratory Data Analysis

Fourteen people work in a government staff office. Let age be the statistic of interest associated with each element (a person) in the set. The corresponding data set is $\{51,30,33,46,50,32,54,41,50,51,51,64,53,19\}$ and the rank-ordered data set is $\{19,30,32,33,41,46,50,50,51,51,51,53,54,64\}$. We shall consider this data set to be a sample.

1. Stem-and-Leaf Plot: a vertical histogram using the actual data points:

   Shown to the right is an example using the above data set:

$$\begin{array}{l} 6:4 \\ 5:0,0,1,1,1,3,4 \\ 4:1,6 \\ 3:0,2,3 \\ 2: \\ 1:9 \end{array}$$

2. Measures of Location: $\bar{x} = 44.5$, $\tilde{x} = 50$, $M = 51$

3. Rules for Calculating Percentile Ranks (once the data set is ordered):

   a) Compute a position index $I = (P/100) \cdot N$ where P is the percentile of interest and N is the total number of data points.
   b) If I is not an integer, round up.
   c) If I is an integer, the P^{th} percentile is the average of the data of the data values in positions I and $I+1$. Note: this process breaks down for the 100^{th} percentile.

4. Quartile Definitions: $Q_1 = P^{25}$, $Q_2 = P^{50} = \tilde{x}$, $Q_3 = P^{75}$

   In our example, the position index for the 25^{th} percentile is $I = (25/100) \cdot 14 = 3.5$. Round up to obtain position 4 in the rank-ordered data set. The data value in position 4 is 33. Hence $Q_1 = P^{25} = 33$. Likewise $Q_2 = P^{50} = \tilde{x} = 50$, $Q_3 = P^{75} = 51$.

5. Five-Number Summary (for a data set where $N \geq 5$).
   The five-number summary is the derived rank-ordered data set $\{x_S, Q_1, x, Q_3, x_L\}$.
   In our example, the five-number summary is $\{19,33,50,51,64\}$

6. Inter-quartile Range: $R_{IQ} = Q_3 - Q_1$

9) Sample Exams

9.1 1885 Entrance High School Exam: Arithmetic

If you wanted to *enter* Jersey City High School back in 1885—eighteen years before the Wright Brother's first powered flight at Kittyhawk, North Carolina!—you first had to pass an entrance exam covering five basic academic disciplines: arithmetic, geography, United States history, grammar, and algebra. The ten questions below comprise the 1885 arithmetic exam.

1. If a 60-days note of $840.00 is discounted at 4.5% by a bank, what are the proceeds?
2. The interest on $50.00 from 1 March to 1 July is $2.50. What is the annual simple interest rate?
3. The mason work on a building can be finished by 16 men in 24 hours, working 10 hours a day. How long will it take 22 men working 8 hours a day?
4. By selling goods at 12.5% profit, a man clears $800.00. How much did they cost? For how much were they sold?
5. What is the cost of 83 pounds of sugar at $98.50 a ton?
6. A merchant sold some goods at a 5% discount for $18,775.00 and still made a 10% profit. What did the merchant pay for the goods?
7. Find the sum of $\sqrt{16.7281}$ and $\sqrt{.721\frac{1}{4}}$
8. Find $(.37-.095)\div(.00025)$ .Express the result in words.
9. A requires 10 days and B 15 days to paint a house. How long will it take A and B together to paint the house?
10. A merchant offered some goods for $1170.90 cash, or $1206 payable in 30 days. Find the simple interest rate.

9.2 1885 High School Entrance Exam: Algebra

Below is the algebra portion from the same 1885 Jersey City High School entrance exam.

1. Define algebra, algebraic expression, and polynomial.
2. Simplify: $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$.
3. Find the product of $3+4x+5x^2-6x^3$ & $4-5x-6x^2$.
4. Write a homogeneous quadrinomial of the third degree.
5. Express the cube root of $10ax$ in two ways.
6. Find the prime factors of x^4-b^4 and x^3-1.
7. Find both the sum and difference of the two expressions $3x-4ay+7cd-4xy+16$ & $10ay-3x-8xy+7cd-13$.

8. Divide the expression $6a^4 + 4xa^3 - 9(ax)^2 - 3ax^3$ by the expression $2a^2 + 2ax - x^2$ and check.

9. Find the G.C.D. of $6a^2 + 11ax + 3x^2$ & $6a^2 + 7ax - 3x^2$.

10. Divide $\dfrac{x^2 - 2xy + y^2}{ab}$ by $\dfrac{x-y}{bc}$ and give the answer in its lowest terms.

9.3 1947 High School Exit Exam: Algebra

In 1947, the United States Air Force officially came into being. Below is the algebra portion of a Canadian high school exit exam from the same year. A score of 80% was required to pass. How do you score in 2003?

1. Prove: $\log_a N^p = p\log_a N$.

2. Plot the graphs of $y = 3x^2 - x^3$ and $y = 3x + 7$ on the same set of axis for the interval $-1 \le x \le 4$. Prove that $y = x^3 - 3x^2 + 3x + 7$ has one real root and find it.

3. If $\dfrac{x}{y}$ varies as $(x+y)$ and $\dfrac{y}{x}$ varies as $x^2 - xy + y^2$, show that $x^3 + y^3$ is a constant.

4. Prove: $a + (a+d) + (a+2d) + ... = \dfrac{n}{2}\left(2a + [9n-1]d\right)$.

5. If $P_n^5 = 90P_{n-2}^3$, find the value of n.

6. How many even numbers of four digits can be formed with the numerals 2, 3, 4, 5, 6, if no numeral is used more than once in each number?

7. If m and n are the roots of the equation $ax^2 + bx + c = 0$, prove that $m + n = -\dfrac{b}{a}, mn = \dfrac{c}{a}$.

8. One root of the equation $x^2 - (3a+2)x + 12 = 3$ is three times the other. Find the value of a.

9. Expand $\dfrac{1}{(1-3x)^2}$ to 4 terms in ascending powers of x.

10. Show that when higher powers of x can be neglected, $\dfrac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1 + x + \sqrt{1+x}}$ is approximately $1 - \dfrac{5}{6}x$.

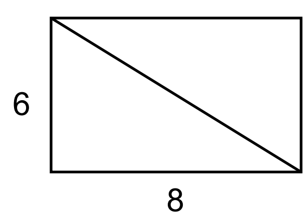
9.4 A Very Basic Pre- Algebra Exam for 2003

Answers Here

1. Add: 21.3 + 10.6 + 24.2 ________________
2. Subtract: 12.9 – 19.7 ________________
3. Evaluate: 9 + 5x6 ________________
4. Simplify: 3m + 8m ________________
5. Simplify: $2ab + 3a^2b + 7ab$ ________________
6. Solve: 2y + 5y = 42 ________________
7. Add: 71/2 + 112/3 ________________
8. Write 2000 as a product of prime numbers ________________
9. Write 2001 as a product of prime numbers ________________

For the rectangle shown in the figure below, find:

10. Area ________________
11. Perimeter ________________
12. Length of the Diagonal ________________



13. 85% of 120 equals what number? ________________

9.5 An Elementary Algebra Exam for 2003

1.Solve and check the following four equations.

a) $5[3(7-t)-4(8+2t)]-20=-6[2(6+3t)-4]$

b) $2(y+1)y^2+12(y+1)y=-10(y+1)$

c) $(x+3)(2x-1)=9$

d) $x^2+5x+6=0$

2. Multiply the expression $(2x-3y)^4$.

3. The length of one leg of a right triangle is 7 ft longer than the other. The length of the hypotenuse is 13 ft. Find the perimeter and area of the triangle.

4. Simplify the expression $\dfrac{3x^2-2x-1}{x^2-3x+2}$.

5. Divide and simplify the expression $\dfrac{3x^2+11x-4}{4x^2+9x-9} \div \dfrac{6x^2+x-1}{4x^2+5x-6}$.

6. Add and simplify the expression $\dfrac{x}{x^2+11x+30}+\dfrac{-5}{x^2+9x+20}$.

7. Simplify the complex rational expression $\dfrac{a^{-1}+b^{-1}}{\dfrac{a^2-b^2}{ab}}$.

9.6 An Advanced Algebra Exam for 2003

The emphasis is on the solving of equations and word problems.

1. Solve and check the following equations.

a) $\frac{1}{w} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ for w b) $2\sqrt{x-3} + \sqrt{3x-5} = 8$ c) $x^3 + 8 = 0$

d) $\sqrt{\frac{x-3}{x-4}} - \frac{1}{\sqrt{x+4}} = \frac{1}{2}$ e) $\frac{x^2}{x^2-5x+6} = \frac{2}{x-2} + \frac{6}{(x-2)(x-3)}$

f) $\sqrt[4]{5x^2-6} = x$ g) $\frac{\sqrt[3]{x}+1}{\sqrt{x+1}} = \frac{\sqrt[3]{x}-1}{\sqrt{x+1}-2}$ h) $\sqrt[3]{x^2} - 3\sqrt[3]{x} + 2 = 0$

2. Evaluate the following two expressions. *Hint: If your calculator overflows, try logarithms!*

a) $\frac{(7.25)^{1359} \times \sqrt{(7.14)^{13.5}}}{(3.39)^{1481}}$ b) $\frac{(9.2)^{545} \times (5.33459)^{24.79}}{(4.15)^{934} \times \sqrt[7]{519.395}}$

3. A rectangular solid has the length of each side increased by the same amount in order to double the volume. Find the revised dimensions if the original dimensions are 3 by 4 by 5 cm.

4. A pyramid with square base has a volume of 100 cubic centimeters. The slant height is 1.2 times the altitude. Find the dimensions of the pyramid.

5. Find six right triangles where all three sides are of integral length, given that one side is equal to 48 and the perimeter is less than 300.

6. A plane left an airfield to fly to a destination 1860 miles away. After flying at a certain airspeed for 600 miles, the wind changed increasing the airspeed of the plane by 40 mph. This reduced the time of the trip by 45 minutes. What was the original airspeed of the plane?

7. A dealer bought a shipment of shoes for $480.00. He sold all but 5 pairs at a profit of $6.00 per pair, thereby making a total profit of $290.00 on the shipment. How many pairs of shoes were in the original shipment?

8. The boundary fence enclosing a rectangular plot of ground is 128 yards long. The area of the enclosed plot is 960 square yards. What are the dimensions—height, width, and diagonal—of the enclosed rectangular plot?

9. Two train stations A and B are 300 miles apart and in the same time zone. At 5AM a passenger train leaves A for B and a freight train leaves B for A. The two trains meet at a point 100 miles from B. Had the speed of the passenger train been 10 mph faster, it would have reached B 9 hours before the freight train reached A. How fast was each train traveling?

9.7 A Mathematics of Finance Exam for 2003

1. Find the final value of $250,000.00 deposited at 7% interest for a period of 10 years if **A**) the compounding is yearly, **B**) quarterly, **C**) monthly, and **D**) continuous.

2. Find the effective interest rate R_{eff} for each of the four cases **A**, **B**, **C**, & **D** in 1).

3. You put $5000.00 down on a car costing $30,000.00 and finance the rest over a period of 6 years at 5% simple interest. **A**) What is your monthly payment? **B**) What is the total payback assuming the loan is kept to completion? **C**) How much of this payback is interest?

4. Fill in the following table comparing three potential mortgages with $P = \$230{,}000.00$.

Fixed Rate Mortgage with $P = \$230{,}000.00$				
Years	r	M_{Pymt}	Total A	Total I
$T = 30$	6.50%			
$T = 20$	6.25%			
$T = 15$	5.50%			

5. You start an Individual Retirement Account (IRA) at age 25 by investing $\$7000.00$ per year in a very aggressive growth fund having an annual rate of return that averages 13%. Five years later, you roll the proceeds from this fund into a blue-chip growth fund whose average long-term annual-rate-of-return is 9%. You also increase your annual contribution to $9000.00 and faithfully continue this to age 69.

> **A**) Assuming continuous and steady interest rates, project the face value of your total investment when you reach age 69.
>
> **B**) What is the present value of total projected in part **A**) if inflation holds at a steady rate of $i = 3\%$ throughout the 44-year period?
>
> **C**) What is the present value of the monthly payment associated with an annuity bought at age 69 with the total in **B**). Assume the annuity pays a fixed 4% and is amortized at age 100.
>
> **D**) If you actually lived to be age 100, what would be the present value of the final annuity payment if the inflation rate remains relatively constant at 3% throughout this 75-year period?

6. You start an IRA at age 25 with a monthly deposit of $200.00 taken from your paycheck. The account pays 8% annually, and you keep this practice up until the age of 35. At 35, you roll the account over into a more risky, high performing IRA paying 12% annually. You also increase your monthly deposits to $350.00. You faithfully continue this practice until the age of 50. At 50, you have to stop the $350.00 monthly deposit due to your children's college expenses and make no new deposits in the account until age 57. Also from age 50 to 57, the account's average annual rate of return drops to 9%. At age 57 you really get serious when the youngest child finally obtains that coveted college diploma! You transfer all your retirement funds to a more aggressive account paying an average of 14% annually and start making a monthly deposit of $600.00. What is the value of your accumulated funds at age 65? Assume continuous interest throughout.

10) Forms: Polynomial Worksheet

General Form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

My Polynomial: $P(x) =$ ____________________________________

Initial Observations

1. Degree of polynomial? _____________
2. Degree odd or even? ______________
3. Number of roots: __________________
4. Are coefficients real? _______________
5. Are coefficients rational? ____________

Descartes Rule of Signs

Variations in $P(x)$: ___ # in $P(-x)$: ____

Combo	Pos	Neg	Com

Rational Root Possibility List

Entry: p divides a_0; q divides a_n.

p :___________________________________

____

q :___________________________________

____

$\frac{p}{q}$:__________________________________

____

____

____

Root Trial Matrix

x	$P(x)$	Root?	Bound?	Sign

Observations on Root Trials

Successive Factorizations

1: $P(x) =$ ____________________________

2: $P(x) =$ ____________________________

3: $P(x) =$ ____________________________

4: $P(x) =$ ____________________________

5: $P(x) =$ ____________________________

11) Engineering Conversion Factors

11.1 Master Table of Conversions

To Convert	To	Multiply By
acres	ft^2	43560
acres	m^2	4046.9
acres	rods	160
acres	hectares	.4047
acre feet	barrels	7758
acre feet	m^3	1233.5
angstrom, å	cm	10^{-8}
angstrom, å	nm	0.1
astronomical unit, AU	cm	1.496E13
astronomical unit	Tm	0.1496
atmospheres (atm)	feet of water	33.94
atmospheres	in of Hg	29.92
atmospheres	mm of Hg	760
atmospheres	psi	14.7
bar	atm	.98692
bar	dyne cm^{-2}	10^6
bar	psi (lb in^{-2})	14.5038
bar	mm Hg	750.06
bar	MPa	10^{-1}
barrel (bbl)	ft^3	5.6146
barrel	m^3	.15898
barrel	gal (US)	42
barrel	liter	158.9
BTU	Canadian BTU	1.000418022
BTU	ISO BTU	1.000527124
BTU	cal	251.996
BTU	erg	$1.055055853 * 10^{10}$
BTU	joule	1054.35
calorie (gm) (cal)	joule	4.184
centimeter (cm)	inch	0.39370
cm	m	10^{-2}
darcy	m^2	9.8697E-13
dyne	g cm s^{-2}	1
dyne	Newton	10^{-5}
erg	cal	2.39006E-8

To Convert	To	Multiply By
erg	dyne cm	1
erg	joule	10^{-7}
fathom (fath)	ft	6
feet (ft)	in	12
feet	m	0.3048
furlong	yd	220
gallon (US) (gal)	in^3	231
gallon	liter	3.78541
gallon (Imp.) (gal)	in^3	277.419
gallon	liter	4.54608
gamma	gauss	10^{-5}
gamma	Tesla	10^{-9}
gauss	Tesla	10^{-4}
gram (g)	pound	0.0022046
gram	kg	10^{-3}
hectare	acre	2.47105
hectare	cm^2	10^8
horsepower	W	745.700
inch (in)	cm	2.54
inch (in)	mm	25.4
joule (J)	erg	10^7
joule	cal	0.239006
kilogram (kg)	g	10^3
kilogram	pound	2.20462
kilometer (km)	m	10^3
kilometer	ft	3280.84
kilometer	mile	0.621371
kilometer hr^{-1} (kph)	mile hr^{-1} (mph)	0.621371
kilowatt	hp	1.34102
knot	mph	1.150779
liter	cm^3	10^3
liter	gal (US)	0.26417
liter	in^3	61.0237
meter	angstrom	1×10^{10}
meter	ft	3.28084
micron	cm	10^{-4}
mile	ft	5280
mile	km	1.60934
mm Hg	dyne cm^{-2}	1333.22

To Convert	To	Multiply By
Newton	dyne	10^5
Newton	pound (lbf)	0.224809
Newton-meter (torque)	foot-pound-force	.737562
ounce	lb	0.0625
Pascal	atmospheres	$9.86923 * 10^{-6}$
Pascal	psi	$1.45 * 10^{-4}$
Pascal	torr	$7.501 * 10^{-3}$
pint	gallon	0.125
poise	$g\ cm^{-1}\ s^{-1}$	1
poise	$kg\ m^{-1}\ s^{-1}$	0.1
pound (lbm)	kg	0.453592
pound (lbf)	Newton	4.4475
rod	feet	16.5
quart	gallon	0.25
stoke	$cm^2\ s^{-1}$	1
slug	kg	14.594
tesla	gauss	10^4
Torr	Millibar	1.333224
Torr	Millimeter of Hg	1
ton (long)	lb	2240
ton (Metric)	lb	2205
ton (Metric)	kg	1000
ton (short or net)	lb	2000
ton (short or net)	kg	907.185
ton (short or net)	ton (Metric)	.907
watt	$J\ s^{-1}$	1
yard	in	36
yard	m	0.9144
year (cal)	days	365.242198781
year (cal)	s	3.15576×10^7

11.2 Table of Basic Engineering Units

Length	meter	m
Time	second	s
Mass	kilogram	kg
Temperature	Kelvin	K
Electrical Current	ampere	A

11.3 Table of Derived Engineering Units

Force	Newton	N	kg m s^{-2}
Energy	joule	J	kg m^2 s^{-2}
Power	watt	W	kg m^2 s^{-3}
Frequency	hertz	Hz	s^{-1}
Charge	coulomb	C	A s
Capacitance	farad	F	C^2 s^2 kg^{-1} m^{-2}
Magnetic Induction	Tesla	T	kg A^{-1} s^{-2}

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